TTIC 31260 - Algorithmic Game Theory (Spring 2024)

Lecturer: Avrim Blum | Lecture 13: May 1, 2024

Fair Division and Cake Cutting

1 Overview

Today we'll be talking about the concept of fair division of some collection of resources, which has many important applications in economics and the social sciences. The metaphor we'll be using is how to fairly divide a cake (though some of the protocols won't necessarily so practical for actually cutting cake...). A good book on this kind of material is Brams and Taylor, "Fair Division".

2 Definitions and Assumptions

Definition 1 *A* protocol *is a set of instructions for n participants. A protocol is like an algorithm except that the participants may choose to not follow the protocol if they wish.*

Assumptions:

1. Each participant i has a personal non-negative value function v_i over portions of the cake, with the entire cake having value 1. Formally, the cake is the interval $[0, 1]$ and v_i should be a non-negative bounded density function that integrates to 1. Intuitively, we think of the cake as being not necessarily uniform (may have chocolate, vanilla, frosting, etc) and some parts of the cake (e.g., chocolate) may be more valuable to one person than another.
2. The cake is a divisible good (can be cut into disjoint intervals in any way we like). Each participant i should be assigned a finite union of these intervals, and their value is the integral of their v_i over that union. (So, valuations are additive).
3. Each person follows the *observable* aspects of the protocol. For example, if the protocol calls for participant #1 to cut the cake into 3 pieces of equal value to them, they will indeed divide it into 3 pieces, not 2 or 4. However, the pieces may not truly have equal value to them (in which case they are not following protocol). So, e.g., we are not going to worry about someone stealing the entire cake in the middle of the protocol....

Definition 2 *A cake cutting protocol is* fair *if each participant i who follows protocol is guaranteed to receive at least $1/n$ of the cake according to their own personal value system v_i.*

Note: it is possible for each participant to have *more* than $1/n$ of the cake - can you see why?

Definition 3 *A cake cutting protocol is* envy-free *if no participant who follows protocol would ever want to switch their allotment with that given to any other participant. That is, if each person j is given subset S_j, then for all i, j we have $v_i(S_i) \geq v_i(S_j)$.*

3 Two players

Say we have two players. If player 1 divides the cake into two pieces and decides which one is for themselves and which is for player 2, would that be fair? No. what would be better?

Cut-and-choose: Player 1 divides the cake into two pieces of equal value according to v_1. Player 2 chooses the piece of higher value according to v_2.

Claim 1 *Cut-and-choose is fair and envy-free.*

Proof: Pretty obvious: player 1 is indifferent between the two pieces (so has no envy and gets value at least 1/2). Player 2 takes their favorite piece so has no envy. ■

4 n players

Claim 2 *For any number of players n, if an allocation is envy free then it is fair. But an allocation might be fair but not envy-free.*

Proof: Let's call the pieces $S_1, ..., S_n$. In the first direction, for each player i, if the allocation is envy-free then $v_i(S_i) \geq v_i(S_j)$ for all j. But we know $\sum_j v_i(S_j) = 1$. So, $v_i(S_i) \geq 1/n$. An example of an allocation that is fair but not envy free would be if a cake is part vanilla, part chocolate, and part carrot; player 1 gets the vanilla, player 2 gets the chocolate, and player 3 gets the carrot. Players 1 has value 1/3 on the vanilla, 2/3 on the chocolate, and 0 on the carrot, but players 2 and 3 value all parts equally. ■

Question: Consider the following protocol for 3 players: person 1 splits the cake into three equal parts (according to v_1), then person 2 chooses their favorite, then person 3 chooses their favorite from what's left, and then person 1 gets the last piece. Is this fair?

Here is a fair protocol:

Moving knife: A knife is slowly swept across the cake. When the value to you of the first piece reaches $1/n$ according to your own value function, raise your hand. The first person to raise their hand gets that piece (ties broken arbitrarily). The protocol then continues with the rest of the cake.

Why is this fair? (Anyone who gets cake before you gets at most $1/n$ value according to you, so there is always at least $1/n$ left for you.

Is it envy-free? No. Why not?

Here is a discrete version of the same protocol:

Moving knife, version 2: Each player i submits a bid $b_i \in [0, 1]$ for where the knife should cut, that is, $v_i([0, b_i]) = 1/n$. The lowest bidder wins their piece of cake, and the protocol then continues with the rest of the players and the remainder of the cake.

Is this incentive-compatible? No. Why not?

5 Envy-free cake cutting for 3 players (Selfridge and Conway)

Here is an envy-free protocol for 3 players. We will do this in two stages. In the first stage, we will create an envy-free division but where there is some unallocated extra portion. Then we'll figure out a way to divide that extra portion to keep things envy-free.

Stage 1: Divide cake with some leftovers. We do this as follows:

1. Person 1 divides the cake into three equal portions A, B, C according to v_1. Without loss of generality, say that $v_2(A) \geq v_2(B) \geq v_2(C)$.
2. Person 2 trims A, removing some portion Z and leaving A' remaining, so that $v_2(A') = v_2(B)$. For now, let's put Z aside.
3. Person 3 takes their favorite piece out of A', B, C.
4. Person 2 takes their favorite piece from the two still left (we're not considering Z) with the condition that they must take A' if it still remains.
5. Person 1 takes the remaining piece (we're still not considering Z).

Claim 3 *The state at the end of Stage 1 is envy free.*

Proof: Let's first consider player 3. It's envy-free for player 3 because they pick first. Now let's consider player 2: it is envy-free to them because they have two equally-favorite pieces among $\{A', B, C\}$, one of which is A'. So (assuming they followed protocol) they are guaranteed to get one of their favorite pieces. Finally, it is envy free for player 1 because they split things equally in the first place, so the only bad piece for them is A'. But they are guaranteed not to get A'. ■

Stage 2: Divvying up Z. At first glance, this seems to be back to the original problem. However, there is property we can now exploit: player 1 is guaranteed not to envy the player who got A' in Stage 1 (do you see why?). The Stage-2 protocol proceeds as follows:

1. Whichever of player 2 or 3 did *not* get A' divides Z into three pieces Z', Z'', Z''' of equal value to them.
2. Whichever of player 2 or 3 *did* get A' chooses first.
3. Player 1 chooses second.
4. The player who divided up Z chooses last.

Claim 4 *The full protocol is envy-free.*

Proof: The easy case is the player who chose first doesn't envy any of the others because they chose first. It's also envy-free to player 1 because they choose before the only player they might envy. Finally, it is envy-free to the divider because they divided the cake into three equal pieces. ■

Note: it's crucial that values are additive so that if you don't have envy in Stage 1 and you don't have envy in Stage 2 then you don't have envy overall.